Exponential and strong ergodicity for time-changed symmetric stable processes

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More results in:

- Tao Wang. Exponential and strong ergodicity for one-dimensional time-changed symmetric stable processes. *Bernoulli*, **29**, no.1 (2023), 580–596.

Ergodicity

- Let $X := (X_t)_{t\geqslant 0}$ be a ergodic Markov process with transition semigroup $(P_t)_{t\geqslant 0}$, generator L and stationary distribution π.
- Exponential ergodicity:

$$\|P_t(x,\cdot)-\pi\|_{\mathrm{Var}} \leqslant C(x)\mathrm{e}^{-\lambda_1 t}.$$

$\hookrightarrow$Optimal convergence rate in exponential ergodicity: spectral gap (reversible case)

$$\lambda_1 = \inf\{\langle -Lf, f\rangle_\pi : f \in \mathscr{F}, \pi(f^2)=1, \pi(f)=0\}.$$

- Strong ergodicity:

$$\sup_x \|P_t(x,\cdot)-\pi\|_{\mathrm{Var}} \leqslant C\mathrm{e}^{-\kappa t}.$$

$\hookrightarrow$Optimal convergence rate in strong ergodicity:

$$\kappa = -\lim_{t\to\infty}\frac{1}{t}\log\sup_x \|P_t(x,\cdot)-\pi\|_{\mathrm{Var}}.$$

Motivation from one-dimensional diffusions

- Let X be a diffusion process on $[0,\infty)$ with reflecting boundary 0 and generator
$$\mathcal{A} = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x},$$
where $a > 0$ and a, b is locally bounded.
- Define $C(x) = \int_0^x b/a$. Then speed measure: $\mu([x,y]) = \int_x^y \mathrm{e}^C/a$ and scaling function: $\varphi(x) = \int_0^x \mathrm{e}^{-C}$.
- Only using speed measure μ and scaling function φ, we can describe ergodicity and functional inequalities explicitly (see [Chen 05: Eigenvalues, Inequalities, and Ergodic Theory]).

Eleven criteria for one-dimensional diffusions

Speed measure: $\mu([x,y]) = \int_x^y \mathrm{e}^C/a$; scaling function: $\varphi(x) = \int_0^x \mathrm{e}^{-C}$

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]\varphi'(x)\mathrm{d}x = \infty \quad (*)$
Recurrence	$\varphi(\infty) = \infty$
Ergodicity	$(*)\&\mu[0,\infty) < \infty$
Exponential ergodicity L^2-exp.convergence	$(*)\& \sup_{x>0} \mu[x,\infty)\varphi(x) < \infty$
Discrete spectrum	$(*)\& \lim_{n\to\infty} \sup_{x>n} \mu[x,\infty) \int_n^x \varphi'(x)\mathrm{d}x = 0$
Log-Sobolev inequality Exp. convergence in entropy	$(*)\& \sup_{x>0} \mu[x,\infty) \log\left[\mu[x,\infty)^{-1}\right] \varphi(x) < \infty$
Strong ergodicity L^1-exp.convergence	$(*)\& \int_0^\infty \mu[x,\infty)\varphi'(x)\mathrm{d}x < \infty$
Nash inequality	$(*)\& \sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu}\varphi(x) < \infty$

A new viewpoint to describe ergodicty by harmonic function and reversible measure

- Speed measure $\mu([x,y]) = \int_x^y \mathrm{e}^C/a \Leftrightarrow$ reversible measure;
- Scaling function $\varphi(x) = \int_0^x \mathrm{e}^{-C} \Leftrightarrow$ harmonic function (killed upon 0).
- Describe ergodicity and functional inequalities by reversible measure μ and harmonic function φ.

A new viewpoint to describe ergodicty by harmonic function and reversible measure

- A natural idea is to extend these problems to stable jump diffusion Y, which has generator

$$L = a(x)\Delta^{\alpha/2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x},$$

where $a > 0$ is continuous, and $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the fractional Laplacian which has the following expression:

$$\Delta^{\alpha/2} f(x) = \int_{\mathbb{R}\setminus\{0\}} \left(f(x+z) - f(x) - f'(x) z \mathbf{1}_{\{|z|\leqslant 1\}}\right) \frac{C_{1,\alpha}}{|z|^{1+\alpha}} \mathrm{d}z.$$

Stable jump diffusions: related works

Under some mild assumptions, there are some sufficient conditions for stable jump diffusions:

- By coupling methods: SDE driven by additive stable noise [Luo-Wang, 2019, SPA];
- By Meyn-Tweedie's condition: time-changed symmetric stable processes [Chen-Wang, 2014, SPA]; supercritical SDEs driven by stable processes [Zhang-Zhang, 2023, Bernoulli]; SDE driven by symmetric stable noise [Huang-Wang, 2023+, preprint].
- Problem 1: find criteria which are similar to the 1-D diffusions, for strong ergodicity and exponential ergodicity of this process by using reversible measures and harmonic functions.
- Problem 2: give the estimates for convergence rates, including spectral gap and convergence rate in strong ergodicity.

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When is the stable jump diffusion reversible?

- Reversible if and only if $b \equiv 0$[1] (it is a corollary from [Kühn-Schilling, 2019, JFA]), which implies that $L = a(x)\Delta^{\alpha/2}$.
  - Reversible measure: $\pi(\mathrm{d}x) = a(x)^{-1}\mathrm{d}x$.
  - Harmonic function (killed upon 0): $h_0(x) = (\omega_\alpha/2)|x|^{\alpha-1}$ with $\omega_\alpha = -(\cos(\pi\alpha/2)\Gamma(\alpha))^{-1} > 0$.
- Associated process: The process Y with generator $a(x)\Delta^{\alpha/2}$ can be described by
  - time-changed symmetric stable process: $Y_t := Z_{\zeta_t}$, where $\zeta_t := \inf\left\{s > 0 : \int_0^s a(Z_u)^{-1}\,\mathrm{d}u > t\right\}$ and Z is a symmetric stable process;
  - stochastic differential equation: let $\sigma = a^{1/\alpha}$ (positive, continuous). Y_t is the unique (possibly exploding) weak solution [Döring-Kyprianou, 2020, AoP]:

$$\mathrm{d}Y_t = \sigma(Y_{t-})\,\mathrm{d}Z_t.$$

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Time-changed symmetric stable processes: related works

- Recurrence and ergodicity: a time change does not change its transience and recurrence.
  A d-dimensional stable process is recurrent iff $\alpha \geqslant d \Rightarrow d = 1$ and $\alpha \in [1, 2)$.
  So multidimensional time-changed stable process can not be ergodic, and

$$\text{recurrent iff} \quad 1 \leqslant \alpha < 2 \begin{cases} \text{pointwise recurrent :} & 1 < \alpha < 2 \\ \text{neighborhood recurrent :} & \alpha = 1 \end{cases}$$

and

$$\text{ergodic iff} \quad 1 \leqslant \alpha < 2 \quad \text{and} \quad \pi(\mathbb{R}) < \infty.$$

Time-changed symmetric stable processes: related works

- From now on, assume that $1 < \alpha < 2$.

Theorem (Chen-Wang, 2014, SPA)

If there exists $\gamma > 1$, such that $\lim_{|x|\to\infty} \sigma(x)/|x|^\gamma > 0 \Rightarrow$ strongly ergodic; if $\lim_{|x|\to\infty} \sigma(x)/|x| > 0 \Rightarrow$ exponentially ergodic/Poincaré inequality.

- Problem 1: find criteria which are similar to the 1-D diffusions;
- Problem 2: convergence rates.

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if $\lim_{|x|\to\infty} \sigma(x)/|x| > 0 \Rightarrow$ *exponentially ergodic/Poincaré inequality.*

- Problem 1: find criteria which are similar to the 1-D diffusions;
- Problem 2: convergence rates.

Main result: exponential ergodicity

Theorem (W. 2023, Bernoulli)

Y is exponentially ergodic if and only if

$$\delta := \sup_x |x|^{\alpha-1} \int_{\mathbb{R}\setminus(-|x|,|x|)} \sigma(y)^{-\alpha} \mathrm{d}y < \infty.$$

Furthermore,

$$\lambda_1 \geqslant (4\omega_\alpha \delta)^{-1},$$

where $\omega_\alpha := -(\cos(\pi\alpha/2)\Gamma(\alpha))^{-1} > 0.$

- Comparison: diffusion: $\sup_{x>0} \mu[x,\infty)\varphi(x) < \infty$; time-changed stable: $\sup_{x>0} \pi((-x,x)^c)h_0(x) < \infty.$
- When $\alpha \to 2$, the case is reduced to time-changed Brownian motion on $\mathbb{R}$. In this case, the process is exponentially ergodic if and only if $\delta := \sup_{x>0} x \int_{\mathbb{R}\setminus(-x,x)} \sigma(z)^{-2}\mathrm{d}z < \infty$, and $\lambda_1 \geqslant (4\delta)^{-1}$.
- (Corollary) [Chen-Wang, 2014, SPA] Y is exponentially ergodic if $\liminf_{|x|\to\infty} \sigma(x)/|x| > 0.$

Sketch of proof

- Exponential ergodicity $\Leftrightarrow$ Poincaré inequality $\Leftrightarrow$ the existence of the spectral gap

$$\lambda_1 := \inf\{\mathscr{E}(f,f) : f \in \mathscr{F}, \pi(f^2) = 1, \pi(f) = 0\}.$$

In general, it is difficult to estimate λ_1.

- Upper bound for λ_1 by using $\lambda_0(B)$ [Chen-Wang, 2000, AoP]:

$$\lambda_1 \leqslant \frac{\lambda_0(B)}{\pi(B^c)}, \text{ for any open set } B \text{ with } \pi(B^c) > 0,$$

where

$$\lambda_0(B) = \inf\{\mathscr{E}(f,f) : f \in \mathscr{F}, \pi(f^2) = 1 \text{ and } f|_{B^c} = 0\}.$$

- Lower bound [Chen, 2000, Sci China Ser A]: $\lambda_1 \geqslant \lambda_0 := \lambda_0(\{0\}^c)$.
- Proving exponential ergodicity $\Leftrightarrow$ estimating the local first Dirichlet eigenvalues.

Motivation from 1-D diffusions

[Chen 05: Eigenvalues, Inequalities, and Ergodic Theory] 1-D diffusions:

- Define II-operator: $\mathrm{II}(f)(x) = \frac{1}{f(x)} G_0 f(x)$,
  where $G_0 f(x) := \int_0^x \mathrm{e}^{-C(y)} \left(\int_y^\infty f(z) \mu(\mathrm{d}z) \right) \mathrm{d}y$.
- Variational formula:

$$\lambda_0^{-1} \leqslant \inf_{f \in \mathscr{F}'} \sup_x \mathrm{II}(f)(x), \quad \lambda_0^{-1} \geqslant \sup_{f \in \tilde{\mathscr{F}}} \inf_x \mathrm{II}(f)(x),$$

- $u := G_0 f$ solves the Poisson equation with Dirichlet boundary condition: $-\mathcal{A}u = f$, $u(0) = 0 \Leftrightarrow$ Green operators.

Estimating Dirichlet eigenvalues by Green operators

- Let G^B be the Green operator given by

$$G^B f(x) := \int_B f(y) G^B(x, \mathrm{d}y),$$

$G^B(x, \mathrm{d}y) := \int_0^\infty P_t^B(x, \mathrm{d}y)\mathrm{d}t$, $P_t^B(x, A) := \mathbb{P}_x[Y_t \in A, t < \tau_B]$, $\tau_B := \inf\{t \geqslant 0 : Y_t \notin B\}$.

- $u := G^B f$ solves the Poisson equation $-\mathcal{L}^B u = f, \quad u|_{B^c} = 0$ ([Oshima, 2013]): for any $u \in \mathscr{F}^B$,

$$\mathscr{E}(G^B f, u) = \int f u \mathrm{d}\mu.$$

- Define $\mathrm{II}(f)(x) = \frac{1}{f(x)} G^B f(x)$.

Theorem

Variational formula:

$$\inf_{f \in C_b(\mathbb{R})} \sup_{x \in B} \mathrm{II}(f)(x)^{-1} \geqslant \lambda_0(B) \geqslant \sup_{f \in C_b(B)} \inf_{x \in B} \mathrm{II}(f)(x)^{-1},$$

- Problem: how to choose f?

Test functions

- One-dimensional diffusions:
  - Lower bound for λ_0 : by choosing $f=\sqrt{\varphi}:=\sqrt{\int_0^x \mathrm{e}^{-C(t)}\mathrm{d}t}$, (the square root of harmonic function).
  - Upper bound for λ_0: by choosing $f(x)=\varphi(x\wedge x_0)$.
- Conjecture: (recall that $\frac{\lambda_0([-1,1]^c)}{\pi([-1,1])}\geqslant\lambda_1\geqslant\lambda_0:=\lambda_0(\{0\}^c).$)
  - Lower bound for λ_0: by choosing $f=\sqrt{h_0}$, where $h_0(x)=(\omega_\alpha/2)|x|^{\alpha-1}$ is the harmonic function of $P_t^{\{0\}^c}$.
  - Upper bound for $\lambda_0([-1,1]^c)$: by choosing $f(x)=h(x\wedge x_0)$, where $h(x):=\int_1^{|x|}(z^2-1)^{\frac{\alpha}{2}-1}\,\mathrm{d}z$ is the the harmonic function of $P_t^{[-1,1]^c}$ (i.e. $P_t^{[-1,1]^c}h=h$).

Proof of exponential ergodicity

Now by calculations, we have:

- Sufficiency:

Lemma (W. 2023, Bernoulli)

If

$$\delta := \sup_x |x|^{\alpha-1} \int_{\mathbb{R}\setminus(-|x|,|x|)} \sigma(z)^{-\alpha} \mathrm{d}z < \infty,$$

then

$$\mathrm{II}(\sqrt{h_0})(x) \leqslant 4\omega_\alpha \delta.$$

- By variational forumula, $\lambda_1 \geqslant \lambda_0 \geqslant (\mathrm{II}(\sqrt{h_0}))^{-1} \geqslant (4\omega_\alpha\delta)^{-1}$.
- Necessity: [Kyprianou 2018, ALEA Lat. Am. J. Prob. Math. Stat.]

$$G_X^{[-1,1]^c}(x,y) = c_\alpha \left((x-y)^{\alpha-1} h\left(\frac{|xy-1|}{|x-y|}\right) - (\alpha-1)h(x)h(y) \right),$$

where $c_\alpha = \frac{2^{1-\alpha}}{\Gamma(\alpha/2)^2}$ and h is a harmonic function for $P_t^{[-1,1]^c}$:

$$h(x) := \int_1^x (z^2-1)^{\frac{\alpha}{2}-1} \mathrm{d}z.$$

Main result: strong ergodicity

Theorem (W. 2023, Bernoulli)

Y is strongly ergodic if and only if

$$I := \int_{\mathbb{R}} \sigma(x)^{-\alpha}|x|^{\alpha-1}\mathrm{d}x < \infty, \tag{1}$$

furthermore, the convergence rate in strong ergodicity

$$\kappa \geqslant (\omega_\alpha I)^{-1} > 0,$$

where $\omega_\alpha := -(\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right))^{-1} > 0.$

- Comparison: diffusion: $\int_0^\infty \mu[x,\infty)\varphi'(x)\mathrm{d}x = \int_0^\infty \varphi(y)\mu(\mathrm{d}y) < \infty \Leftrightarrow$ time-changed stable: $\int_{\mathbb{R}} h_0(x)\pi(\mathrm{d}x) < \infty$.
- When $\alpha \to 2$: it is strongly ergodic if and only if $\int_{\mathbb{R}} \sigma(x)^{-2}|x|^{\alpha-1}\mathrm{d}x < \infty$ and $\kappa \geqslant I^{-1} > 0$.
- (Corollary) [Chen-Wang, 2014, SPA] Y is strongly ergodic if $\liminf_{|x|\to\infty} \sigma(x)/|x|^\gamma > 0$ for some constant $\gamma > 1$.

Proof of strong ergodicity

- To prove the strong ergodicity of Y, we only need to estimate the uniform bounds for the first moment of hitting time.
- Necessity ([Mao,2002,JAP]): strong ergodicity implies that for any closed set $B \subset \mathbb{R}$ with $\pi(B) > 0$, $\sup_x \mathbb{E}_x \tau_{B^c} < \infty$.
- Sufficiency:

Theorem (Mao-W., 2022, JTP)

Let $M_A := \sup_x \mathbb{E}_x \tau_{A^c}$. *We have*

$$\kappa \geqslant \min\{\lambda_1, M_A^{-1}\}.$$

Note that $\lambda_1 \geqslant \lambda_0$, and $\lambda_0^{-1} \leqslant M_0$ ([Grigor'yan-Telcs, 2012, AoP]). Therefore, $\kappa \geqslant M_0^{-1}$, and $M_0 < \infty$ implies that Y is strongly ergodic.

Proof of strong ergodicity: sufficiency

- $$\mathbb{E}_x \tau_{\{0\}^c} = \int_{\mathbb{R}} G^{\{0\}^c}(x, \mathrm{d}y) = \int_{\mathbb{R}} G_Z^{\{0\}^c}(x,y)\sigma(y)^{-\alpha}\mathrm{d}y,$$

  where $G_Z^{\{0\}^c}(x,y) = -\frac{1}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}\left(|y|^{\alpha-1} + |x|^{\alpha-1} - |y-x|^{\alpha-1}\right).$
- $|y|^{\alpha-1} + |x|^{\alpha-1} - |y-x|^{\alpha-1} \leqslant 2(|x| \wedge |y|)^{\alpha-1}$.
- Thus

$$\sup_x \mathbb{E}_x \tau_{\{0\}^c} \leqslant -\frac{1}{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} \int_{\mathbb{R}} |y|^{\alpha-1}\sigma(y)^{-\alpha}\mathrm{d}y = \omega_\alpha I < \infty.$$

Proof of strong ergodicity: necessity

- Assume that $I := \int_{\mathbb{R}} \sigma(x)^{-\alpha}|x|^{\alpha-1}\mathrm{d}x = \infty$.
- By [Doring-Kyprianou-Weissmann, 2020, SPA],

$$\lim_{x\to\infty} G_X^{[-1,1]^c}(x,y) = K_\alpha h(y),$$

where $K_\alpha = \frac{2c_\alpha(1-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\int_1^\infty \frac{h'(v)}{1+v}\mathrm{d}v < \infty$.

- Therefore,

$$\begin{aligned}
\sup_x \mathbb{E}_x\tau_{[-1,1]^c} &= \sup_x \int_{\mathbb{R}\setminus[-1,1]} G_X^{[-1,1]^c}(x,y)\sigma(y)^{-\alpha}\mathrm{d}y \\
&\geqslant \int_1^\infty \lim_{x\to\infty} G_X^{[-1,1]^c}(x,y)\sigma(y)^{-\alpha}\mathrm{d}y \\
&= \int_1^\infty K_\alpha h(y)\sigma(y)^{-\alpha}\mathrm{d}y \\
&\geqslant \frac{K_\alpha}{(\alpha-1)}\int_1^\infty (y^{\alpha-1}-1)\sigma(y)^{-\alpha}\mathrm{d}y = \infty.
\end{aligned}$$

Example

Consider the polynomial case: $\sigma(x) = (1 + |x|)^{\gamma}$.

- Y is ergodic if and only if $\gamma > 1/\alpha$.
- Y is exponentially ergodic if and only if $\gamma \geqslant 1$. Moreover,

$$\lambda_1 \geqslant \frac{(\alpha\gamma - 1)^{\alpha\gamma}(\alpha(\gamma - 1))^{\alpha(1-\gamma)}}{8\omega_\alpha(\alpha - 1)^{\alpha-1}}$$

  for $\gamma > 1$ and $\lambda_1 \geqslant (\alpha - 1)/8\omega_\alpha$ for $\gamma = 1$.
- Y is strongly ergodic if and only if $\gamma > 1$. Furthermore, $\kappa \geqslant \alpha(\gamma - 1)/2\omega_\alpha$.

Future questions

- Log-Sobolev and Nash inequalities by using Orlicz norm: in preparation.
- Discrete spectrum:
  - For one-dimensional diffusions or birth-death processes: discrete spectrum is equivalent to
    $$\lambda_0((n,\infty)) \to \infty, \quad n \to \infty.$$
  - I have proved that
    $$\lambda_0((-n,n)^c) \to \infty, \quad n \to \infty \tag{2}$$
    is equivalent to $\lim_{n\to\infty} \sup_{x>n} (x^{\alpha-1} - n^{\alpha-1})\pi((-|x|,|x|)^c) = 0.$
  - Problem: whether the super Poincaré inequality is equivalent to (2)?
- $\alpha = 1$, the time-changed Cauchy process (which is not pointwise recurrent): the key method $\lambda_1 \geqslant \lambda_0$ is not valid.
- Ergodicity and functional inequality for reflected time-changed symmetric stable processes (e.g. [Guan-Ma, 2006, PTRF], [Chen-Kim-Kumagai-Wang, 2022, TAMS]) on a bounded domain.

Summary

Criteria for time-changed symmetric α-stable processes $\mathrm{d}Y_t = \sigma(Y_{t-})\mathrm{d}Z_t$ with $\alpha \in (1,2)$ (reversible measure $\pi(\mathrm{d}x) = \sigma(x)^{-\alpha}\mathrm{d}x$, and harmonic function $h_0(x) = (\omega_\alpha/2)|x|^{\alpha-1}$.)

Property	Criterion
Uniqueness	✓
Recurrence	✓
Ergodicity	$\pi(\mathbb{R}) < \infty$.
Exponential ergodicity L^2-exp.convergence	$\sup_x \lvert x\rvert^{\alpha-1}\pi((-\lvert x\rvert,\lvert x\rvert)^c) < \infty$
Discrete spectrum(?)	$\lim_{n\to\infty} \sup_{x>n}(x^{\alpha-1} - n^{\alpha-1})\pi((-\lvert x\rvert,\lvert x\rvert)^c) = 0$
Log-Sobolev inequality(*)	$\sup_x \lvert x\rvert^{\alpha-1}\pi((-\lvert x\rvert,\lvert x\rvert)^c)\log(\pi((-\lvert x\rvert,\lvert x\rvert)^c)^{-1}) < \infty$
Strong ergodicity L^1-exp.convergence	$\int_{\mathbb{R}} \lvert x\rvert^{\alpha-1}\pi(\mathrm{d}x) < \infty$
Nash inequality(*)	$\sup_x \lvert x\rvert^{\alpha-1}\pi((-\lvert x\rvert,\lvert x\rvert)^c)^{(\beta-2)/\beta} < \infty$

QED

Thank you for your attention !